PMM U.S.S.R.,Vol.46.pp.16-21
Copyright Pergamon Press Ltd.1983.Printed in U.K.

0021-8928/83/1 0016 $7.50/0

UDC 531.31:531.36

ON ASYMPTOTIC STOPPING IN THE PRESENCE OF VISCOUS FRICTION*

I. TEREKI and L. KHATVANI

Using Liapunov's second method sufficient conditions are given for asymptotic stopping in stationary and nonstationary mechanical systems with potential forces in the presence of viscous friction with total dissipation. The results are illustrated by example of a material point moving under the action of gravity over a moving surface, as well as by example of a symmetric gyroscope in a gimbal suspension. By definition a mechanical system is asymptotically stopped if with the passing of time the generalized coordinates tend to constants and the generalized velocities tend to zero /1−5/. It turned out /6/ that under very general conditions nonstationary mechanical systems are asymptotically stopped in a neighborhood of a stable equilibrium position under the action of dry friction. It is shown below that in the general case viscous friction now does not cause this phenomenon, and sufficient conditions are given for asymptotic stopping.

1. We consider a holonomic mechanical system with time-dependent constraints, described by the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial q^{\cdot}} - \frac{\partial L}{\partial q} = Q \tag{1.1}$$
$$q = (q_1, \ldots, q_n)^T \in R^n, \ \| q \| = (q_1^2 + \ldots + q_n^2)^{1/2}$$
$$L = L_2 + L_1 + L_0, \ L_2 = {}^1/_2\, (q^{\cdot})^T A\, (t, q)\, q^{\cdot}$$
$$L_1 = B\, (t, q)^T q^{\cdot}, \ L_0 = L_0\, (t, q)$$

A, B, L_0 have continuous partial derivatives on the set

$$\Gamma_q = \{(t, q, q^{\cdot}) : t \in R_+ = [0, \infty), \quad \| q \| < H \leqslant \infty, \ q^{\cdot} \in R^n\}$$

Dissipative and gyroscopic forces whose resultant is denoted $Q = Q\,(t, q, q^{\cdot})$ act on the system; hence, $Q^T q^{\cdot} \leqslant 0$. We assume that system (1.1) admits of the equilibrium position $q = q^{\cdot} = 0$. By K we denote the class of continuous strictly increasing functions $\omega: R_+ \to R_+$ for which $\omega\,(0) = 0$. In /6/ it was shown that if $L_0\,(t, 0) \equiv 0$, $L_0\,(t, q) \leqslant 0$, $\partial L\,(t, q)\,/\,\partial t \geqslant 0$ and a function $\omega \in K$ exists such that the inequality

$$Q^T\,(t, q, q^{\cdot})\, q^{\cdot} \leqslant -\,\omega\,(\| q \|)\, \| q^{\cdot} \| \ (t \in R_+, \ q^{\cdot} \in R^n, \ \| q \| \leqslant H' < H) \tag{1.2}$$

is fulfilled, then the equilibrium position $q = q^{\cdot} = 0$ is stable and any motion for which $\| q\,(t) \| \leqslant H'$ for $t \geqslant t_0$ has a finite limit as $t \to \infty$. Condition (1.2) can be satisfied by dry friction forces /6/ for which, thus, it is typical that the motions asymptotically approach one of the equilibrium positions under specific conditions. Viscous friction forces, i.e., friction forces proportional to the velocity, do not satisfy condition (1.2); of them we can require the fulfillment of the estimate $Q^T q^{\cdot} \leqslant -\,\omega\,(\| q \|)\, \| q^{\cdot} \|^2$ (the dissipation is total).

It happens that a mechanical system exists satisfying the conditions listed above (excepting condition (1.2)), which, under the action of a viscous friction force with total dissipation, has a motion not having a limit as $t \to \infty$ and has a nonstable equilibrium position. An example of such a system is a material point moving under the action of gravity over a surface $z = f\,(r, \varphi)$, where $f\,(r, \varphi) = 0$ $(r \leqslant 1)$ and $f\,(r, \varphi) > 0$ $(r > 1)$, and is found under the action of a viscous friction force with total dissipation. (Here r, φ, z, are cylindrical coordinates, the z-axis is directed opposite to the force of gravity). It is sufficient to show that there exists a function f such that the constraint $z = f\,(r, \varphi)$ admits of the motion $(r\,(t), \varphi\,(t), z\,(t))$ for which

---

*Prikl.Matem.Mekhan.,46,No.1,20-26,1982

$$r(t) \searrow 1,\ \varphi(t) \nearrow \infty,\ z(t) \searrow 0,\ v(t) \to 0 \quad (t \to \infty) \tag{1.3}$$

where $v(t)$ is the magnitude of the velocity at instant t. Then the points of the circle $r = 1$, $z = 0$ are unstable equilibrium positions; moreover, the motions with conditions (1.3) do not have a finite limit as $t \to \infty$. We consider an arbitrary curve $(r(s), \varphi(s), z(s))$ in r, φ, z -space, having the following property: $r(s) \searrow 1$, $\varphi(s) \nearrow \infty$, $z(s) \searrow 0$ $(s \to \infty)$, moreover, $(\beta(s) - e_z)/(r(s) - 1) \to 0$ and $\rho(s) \to 1$ as $s \to \infty$, where $\beta(s)$ is the unit vector of the binormal to the curve, e_z is the unit vector of the z -axis and $\rho(s)$ is the radius of curvature. Let the material point move along this curve as a constraint with a zero initial velocity under gravity in the presence of viscous friction. Simple mechanical consideration shows that the motion indicated satisfies condition (1.3). It remains to find the surface $z = f$ with the properties mentioned, which, as a constraint, admits of this motion. The problem of constructing such a surface is the following. Suppose that along the curve $(r(s), \varphi(s), z(s))$ we are given a sufficiently smooth vector-valued function $\xi(s)$. We need to find a surface passing through the given curve, such that for all values of parameter s the normal to the surface at point $(r(s), \varphi(s), z(s))$ contains the vector $\xi(s)$. The existence of such a surface can be shown by differential-geometric methods.

2. Let us coniser the conditions ensuring the existence of a limit for the generalized coordinates in the presence of viscous friction. For the stationary case it was proved in /3/ that if potential forces do not act and the dissipation is total, then the equilibrium position is stable and the system can be asymptotically stopped. We generalize this result to non-stationary systems under the action of potential forces sufficiently small along the motions.

Lemma. Assume that the functions $f: R_+ \to R_+$, $g: R_+ \to R^n$ are continuously differentiable, the function $w: R_+ \to R_+$ is summable on R_+, while ω: $R_+ \to R_+$ is continuous, nondecreasing and $\omega(r) > 0$ for $r > 0$. We introduce the notation

$$\Omega(r) = \int_0^r \omega(s)\, ds$$

If

$$d/dt\, f(t) \leqslant -\omega(\|g(t)\|)\, \|g^{\cdot}(t)\| + w(t) \quad (t \in R_+)$$

then for any k $(1 \leqslant k \leqslant n)$ the total variation of function $\Omega(|g_k|)$ on the interval $[t', t'']$ is bounded from above by the quantity

$$f(t') - f(t'') + \int_{t'}^{t''} w\, dt \quad (0 \leqslant t' < t'')$$

and the function $g_k(t)$ has a finite limit as $t \to \infty$

Proof. The function $\Omega(|g_k(t)|)$ is absolutely continuous on each interval $[t', t'']$, therefore, its total variation is

$$\int_{t'}^{t''} \left|\frac{d}{dt}\Omega(|g_k(t)|)\right| dt \leqslant \int_{t'}^{t''} \omega(|g_k(t)|)\, |g_k^{\cdot}(t)|\, dt \leqslant -\int_{t'}^{t''} f^{\cdot}(t)\, dt + \int_{t'}^{t''} w(t)\, dt$$

Hence follows the lemma's first assertion. Since $f(t) \geqslant 0$ and function w is summable on R_+, then according to the lemma's first assertion $\Omega(|g_k|)$ is a function of bounded variation on R_+, consequently, it has a finite limit as $t \to \infty$. Then from the condition $\Omega(r) \to \infty$ $(r \to \infty)$ it follows that the function $|g_k(t)|$ is bounded on R_+. If the limit as $t \to \infty$ were not to hold, then the function $\Omega(|g_k(t)|)$ too would not have a limit because $\Omega(r)$ is a strictly increasing function. This leads to a contradiction.

Theorem 1. For system (1.1) assume that the following conditions

1) $\alpha \|q^{\cdot}\|^2 \leqslant (q^{\cdot})^T A(t, q)\, q^{\cdot} \leqslant \beta \|q^{\cdot}\|^2$ $(0 < \alpha, \beta = \text{const})$
2) $(q^{\cdot})^T (\partial A(t, q) / \partial t)\, q^{\cdot} \geqslant -w_1(t) \|q^{\cdot}\|^2$

are satisfied on set $\Gamma_q' = \{(t, q, q^{\cdot}): t \in R_+, \|q\| \leqslant H', q^{\cdot} \in R^n\}$ $(0 < H' < H)$, where the function $w_1(t)$ is nonnegative and summable on R_+;

3) function $w_2(t) = \sup\{\|\partial B(t, q) / \partial t - \partial L_0(t, q) / \partial q\|: \|q\| \leqslant H'\}$ is summable on R_+;
4) dissipation is total, i.e.,

$$Q^T(t, q, q^{\cdot})\, q^{\cdot} \leqslant -\gamma \|q^{\cdot}\|^2 \quad (0 < \gamma = \text{const}) \tag{2.1}$$

Then the equilibrium position $q = q^{\cdot} = 0$ is stable, is asymptotically $q^{\cdot}$-stable, and, for sufficiently small initial values of function $q(t)$, has a finite limit as $t \to \infty$.

Proof. It is well known /5/ that

$$\frac{d}{dt}(L_2 - L_0) = -\frac{\partial L}{\partial t} + Q^T q^{\cdot} \tag{2.2}$$

Consider the Liapunov function $V = 2(L_2)^{1/2}$. On the strength of identity (2.2) and conditions 1) — 4) we have the estimate

$$V^{\cdot}(t, q, q^{\cdot}) \leqslant \left[-\left(\frac{2}{\beta}\right)^{1/2}\gamma + \left(\frac{2}{\alpha}\right)^{1/2} w_1(t)\right] \| q^{\cdot} \| + \left(\frac{2}{\alpha}\right)^{1/2} w_2(t) \tag{2.3}$$

whence by conditions we obtain the inequality

$$V^{\cdot}(t, q, q^{\cdot}) \leqslant \left[-\frac{\gamma}{\beta} + \frac{w_1 t}{\alpha}\right] V(t, q, q^{\cdot}) + \left(\frac{2}{\alpha}\right)^{1/2} w_2(t) \tag{2.4}$$

First of all let us show that for any $\varepsilon > 0$, $t_0 \geqslant 0$ we can find a function $\eta(\varepsilon, t_0) > 0$ such that from the condition

$$\| q(t_0) \|^2 + \| q^{\cdot}(t_0) \|^2 < \eta \tag{2.5}$$

follows $\| q^{\cdot}(t) \| < \varepsilon$ in the interval $t \in [t_0, M)$, where the inequality $\| q(t) \| \leqslant H'$ $(t_0 < M \leqslant \infty)$ is fulfilled. Let the motion $q = q(t)$ $(\| q(t_0) \| < H')$ be given. We consider the function $v(t) = V(t, q(t), q^{\cdot}(t))$. Integrating (2.4) along the motion, we obtain the estimate

$$v(t) \leqslant k(T) + c_1 \int_T^t w_1(s) v(s)\, ds \quad (t_0 \leqslant T \leqslant t < M), \quad k(T) = v(T) + c_2 \int_T^\infty w_2(s)\, ds$$

According to Bellman's lemma /7/ it follows that $v(t) \leqslant c_3 k(T)$ for $t \in [T, M)$. Function w_2 is summable, therefore, there exist $T(\varepsilon)$, $\rho(\varepsilon) > 0$ such that if $T(\varepsilon) < M$ and $v(T(\varepsilon)) < \rho(\varepsilon)$, then $v(t) < \varepsilon (2\alpha)^{1/2}$ when $t \in [T(\varepsilon), M)$. The motions depend on the initial data continuously, therefore, by virtue of condition 1) there exists $\eta(\varepsilon, t_0) > 0$ such that $v(t) < \rho(\varepsilon)$ on the interval $[t_0, T(\varepsilon)]$ when condition (2.5) is fulfilled. Then $v(t) < \varepsilon (2\alpha)^{1/2}$ when $t \in [t_0, M)$. Hence follows the inequality $\| q^{\cdot}(t) \| < \varepsilon$.

Let us now prove that $M = \infty$ for sufficiently small $\| q(t_0) \|^2 + \| q^{\cdot}(t_0) \|^2$, the equilibrium position is $q^{\cdot}$-stable and $q(t)$ has a finite limit as $t \to \infty$. Let γ $(0 < \gamma < H')$ and $t_0 \in R_+$ be given. Assume that

$$\| q(t_0) \|^2 + \| q^{\cdot}(t_0) \|^2 < \eta(1, t_0) \tag{2.6}$$

Then by virtue of (2.3)

$$v^{\cdot}(t) \leqslant -c_4 \| q^{\cdot}(t) \| + w(t) \quad (t_0 \leqslant t < M;\ 0 < c_4 = \text{const}) \tag{2.7}$$

where function w is summable on R_+. By condition 1) there exist δ_1, T $(0 < \delta_1 < \gamma/(2n)$, $t_0 \leqslant T < M)$ such that from $\| q_0 \| < H'$, $\| q_0^{\cdot} \| < \delta_1$ follows

$$V(T, q_0, q_0^{\cdot}) < \frac{c_4 \gamma}{2n} - \int_T^\infty w\, dt$$

The solutions depend continuously in the initial data, therefore, $\delta = \delta(\gamma, t_0)$ exists such that when $\| q(t_0) \|^2 + \| q^{\cdot}(t_0) \|^2 < \delta^2$ the inequality $\| q(t) \|^2 + \| q^{\cdot}(t) \|^2 < \delta^2$ is satisfied on interval $[t_0, T]$. Then $\| q(t) \| < \gamma$ for $t \in [t_0, M)$. In the opposite case there exist numbers t'', k $(T < t'' < M;\ 1 \leqslant k \leqslant n)$ such that $| q_k(t'') | = \gamma / n$. On the other hand, by the lemma it follows from inequality (2.7) that the total variation of function $| q_k |$ is bounded from above by the quantity

$$\frac{1}{c_4} v(T) + \frac{1}{c_4} \int_T^\infty w\, dt < \frac{\gamma}{2n}$$

but this contradicts $| q_k(T) | < \gamma / (2n)$ and $| q_k(t'') | = \gamma / n$. Consequently, $\| q(t) \| < \gamma < H'$ for $t \in [t_0, M)$. From the definition of M it follows that $M = \infty$, i.e., the equilibrium position is stable. Applying the lemma, from (2.7) we get that under sufficiently small initial values the function $q(t)$ has a finite limit as $t \to \infty$.

It remains to prove that the equilibrium position is asymptotically $q^{\cdot}$-stable. As a consequence of (2.7), along the motion we have $v^{\cdot}(t) \leqslant w(t)$ when condition (2.6) is fulfilled. Thus, the nonnegative function

$$v(t) + \int_t^\infty w(s)\,ds$$

does not grow, consequently, it has a finite limit as $t \to \infty$. Since

$$\int_t^\infty w(s)\,ds \to 0$$

as $t \to \infty$, we get that $v(t) \to v^* \geqslant 0$ as $t \to \infty$. In case $v^* > 0$, from (2.7) it would follow that

$$v^{\cdot}(t) \leqslant -c_4 v^*/(2\sqrt{2\beta}) + w(t)$$

i.e., $v(t) \to -\infty$ as $t \to \infty$, but this is impossible. Therefore, $v^* = 0$. But then, by virtue of 1), $\| q^{\cdot}(t) \| \to 0$ as $t \to \infty$. The theorem is proved.

We illustrate this result by the following problem. Let a material point move under the gravity force $(0, 0, -mg)$ in a three-dimensional (x, y, z)-space over a moving surface $z = -\lambda(t)(x^2 + y^2)$ $(\lambda(t) \geqslant 0, \lambda^{\cdot}(t) \leqslant 0, \lambda^{\cdot\cdot}(t) \geqslant 0)$ as a constraint in the presence of viscous friction with total dissipation. If $\lambda(t) = \text{const} > 0$, then the equilibrium position $x = y = 0$ is unstable, while if $\lambda(t) \equiv 0$, it is stable. The question arises on how rapidly the function $\lambda(t)$ should decrease in order that the equilibrium position $x = y = 0$ be stable and the material point be asymptotically stopped. An application of Theorem 1 verifes that the condition

$$\int_0^\infty \lambda(s)\,ds < \infty$$

is sufficient (for example, $\lambda(t) = 1/(1 + t^2)$).

Note. By analyzing the proof of Theorem 1 we can see that if the theorem's conditions are fulfilled on set Γ_q instead of set $\Gamma_{q'}$, then the equilibrium position is stable, is asymptotically $q^{\cdot}$-stable in-the-large and, for any motion, the function $q(t)$ has a finite limit as $t \to \infty$. On the other hand, if instead of condition (2.1) we require the fulfillment of an inequality of more general type $Q^T(t, q, q^{\cdot}) \leqslant -\omega(\|q\|)\|q^{\cdot}\|^2$ with function $\omega \in K$ (see condition (1.2) in the dry friction case), then the theorem's assertions remain valid, excluding the asymptotic $q^{\cdot}$-stability. However, also in this case $\|q^{\cdot}(t)\| \to 0$ $(t \to \infty)$ if the limit of function $\|q(t)\|$ is nonzero.

3. Let us consider stationary mechanical systems under potential, gyroscopic and dissipative forces with total dissipation. Assume that the potential forces act only in certain given directions, i.e., the potential energy depends only on a certain part of the variables, but the equilibrium positions in the subspace of these variables are isolated. The equation of motion has the form (T is the kinetic energy, P is the potential energy)

$$\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot}} - \frac{\partial T}{\partial q} = -\frac{\partial P}{\partial q} + Q, \quad T = {}^1/_2 (q^{\cdot})^T A(q) q^{\cdot} \tag{3.1}$$

Theorem 2. Assume that:

1) a partitioning $q = (q^1, q^2)^T$ of the generalized coordinates $(q^1 \in R^k, q^2 \in R^m; 0 \leqslant k \leqslant n, k + m = n)$ exists such that $\partial P / \partial q^2 = 0$;

2) the solution of the equation $\partial P / \partial q^1 = 0$ are isolated points;

3) the partitioning of the matrix A into blocks, corresponding to the partitioning $q = (q^1, q^2)^T$, has the form $A = \operatorname{diag}\{A^1, A^2\}$ (A^1 is a $k \times k$-matrix, A^2 is an $m \times m$-matrix);

4) the dissipation is total, more precisely, there exist $\gamma_1, \gamma_2 > 0$ such that

$$(Q^1)^T (q^1)^{\cdot} \leqslant -\gamma_1 \|(q^1)^{\cdot}\|^2, \quad (Q^2)^T (q^2)^{\cdot} \leqslant -\gamma_2 \|(q^2)^{\cdot}\|^2$$

Then along each of its bounded motions $(q(t), q^{\cdot}(t))$ the system asymptotically approaches one of its equilibrium positions, i.e., $q(t) \to \text{const}$ and $q^{\cdot}(t) \to 0$ as $t \to \infty$.

Proof. Since the motion is bounded, there exists $c > 0$ such that $P(q(t)) \geqslant c$. Then along the motions the total energy $V = T + P - c$ is positive definite relative to $q^\cdot$ and

$$V^\cdot(q, q^\cdot) = Q^T(q, q^\cdot)\, q^\cdot \leqslant -\gamma \| q^\cdot \|^2 \quad (\gamma = \min(\gamma_1, \gamma_2)) \tag{3.2}$$

By the invariance principle /8/, from estimate (3.2) it follows that $q^\cdot(t) \to 0$ and $q(t)$ tends to a connected set $E \subset R^n$ consisting of the equilibrium positions. Therefore, on the strength of conditions 1) and 2) we get that $q^1(t) \to$ const as $t \to \infty$.

It remains to prove the existence of the limit of function $q^2(t)$. The motion $r(t) = q^2(t)$, $r^\cdot(t) = (q^2)^\cdot(t)$ is, obviously, a solution of the nonstationary Lagrange equation

$$\begin{aligned} &\frac{d}{dt}\frac{\partial L^*}{\partial r^\cdot} - \frac{\partial L^*}{\partial r} = Q^* \\ &L^* = L^*(t, r, r^\cdot) = L_2^*(t, r, r^\cdot) + L_0^*(t, r) \\ &L_2^* = {}^1/_2\, (r^\cdot)^T A^2(q^1(t), r)\, r^\cdot \\ &L_0^* = {}^1/_2\, (q^1)^\cdot(t)^T A^1(q^1(t), r)\, (q^1)^\cdot(t) \\ &Q^* = Q^*(t, r, r^\cdot) = Q^2(q^1(t), r, (q^1)^\cdot(t), r^\cdot) \end{aligned} \tag{3.3}$$

The function $V^* = 2(L_2^*)^{1/2}$ is positive definite and admits of an infinitesimal upper limit relative to $r^\cdot$. Let us estimate the derivative of function V^* relative to system (3.3). Applying identity (2.2), we obtain the formula

$$(L_2^*)^\cdot_{(3.3)} = \frac{\partial L_0^*}{\partial r} r^\cdot - \frac{\partial L_2^*}{\partial t} + Q^* r^\cdot$$

Functions $q(t)$, and $q^\cdot(t)$ are bounded, therefore,

$$\begin{aligned} &(L_2^*)^\cdot_{(3.3)}(t, q^2(t), (q^2)^\cdot(t)) \leqslant \alpha_1 \| (q^1)^\cdot(t) \|^2 \| (q^2)^\cdot(t) \| + \\ &\alpha_2 \| (q^2)^\cdot(t) \|^2 \| (q^1)^\cdot(t) \| - \gamma_2 \| (q^2)^\cdot(t) \|^2 \quad (\alpha_1, \alpha_2 = \text{const}) \end{aligned} \tag{3.4}$$

On the other hand, from (3.2) it follows that

$$\int_0^\infty \| q^\cdot(t) \|^2 dt < \infty \tag{3.5}$$

Using the inequality

$$\alpha_3 \| r^\cdot \|^2 \leqslant L_2^*(t, r, r^\cdot) \leqslant \alpha_4 \| r^\cdot \|^2 \quad (0 < \alpha_3, \ \alpha_4 = \text{const})$$

from (3.4) and (3.5) we obtain the estimate

$$(V^*)^\cdot_{(3.3)}(t, q^2(t), (q^2)^\cdot(t)) \leqslant -\gamma_3 \| (q^2)^\cdot(t) \| + w(t) \quad (\gamma_3 = \gamma_2 / \sqrt{\alpha_4})$$

where the function w is summable on R_+. According to this estimate, the existence of the limit of function $q^2(t)$ follows from the lemma's second assertion. We note that the assertion of Theorem 2 is valid for all bounded motions $(q(t), q^\cdot(t))$ with arbitrary initial data.

4. Let us consider the motion of a symmetric gyroscope in a gimbal suspension, with due regard to the masses of the suspension rings /9/. We assume that the fixed rotation axis of the outer ring of the gimbal suspension is vertical and the rotation axis of the inner ring is horizontal, and that the centers of gravity of the gyroscope and of the inner ring are located on the gyroscope's axis of symmetry. The position of this system can be determined by three Euler angles: the nutation angle θ, the precession angle ψ, and the gyroscope's natural rotation angle φ. We assume that friction forces as well act on the gyroscope besides the force of gravity. Sufficient asymptotic stability condition were given in /9/ for the vertical rotation $\theta = \theta^\cdot = 0$, $\psi^\cdot = \text{const}$, $\varphi^\cdot = \text{const}$ in the presence of any friction with total dissipation. Having studied the influence of dry friction forces, in /6/ the authors proved that if these forces act only in the suspension's axes (see /10/), then each motion asymptotically approaches one of the permanent rotations $\theta = \text{const}$, $\theta^\cdot = \psi^\cdot = 0$, $\varphi^\cdot = \text{const}$. Let us now consider the case of viscous friction, assuming henceforth that the moment of the friction forces relative to the gyroscopes proper axis is zero. After ignoring the cyclic coordinates, we write the equations of motion as

$$\frac{d}{dt}\frac{\partial R}{\partial \theta^\cdot} - \frac{\partial R}{\partial \theta} = -\frac{\partial P}{\partial \theta} + Q_1 \qquad (4.1)$$

$$\frac{d}{dt}\frac{\partial R}{\partial \psi^\cdot} - \frac{\partial R}{\partial \psi} = Q_2$$

$$2R = A(\theta^{\cdot 2} + \psi^{\cdot 2}\sin^2\theta) + A_1\theta^{\cdot 2} + B_1\psi^{\cdot 2}\sin^2\theta + C_1\psi^{\cdot 2}\cos^2\theta + A_2\psi^{\cdot 2} + 2Cr_0\psi^\cdot\cos\theta, \quad P = z_0\cos\theta, \quad r_0 = \varphi^\cdot + \psi^\cdot\cos\theta$$

Here R is the Routh function, Q_1, and Q_2 are friction force moments relative to the suspension axes, r_0 is a cyclic constant. Applying Theorem 2 to system (4.1) (where $q^1 = \theta, q^2 = \psi$) and using the cyclic integral, we obtain the following result. If viscous friction forces act on the gyroscope, whose moments satisfy the inequalities

$$Q_1(\theta, \psi, \theta^\cdot, \psi^\cdot)\theta^\cdot \leqslant -\gamma_1\theta^{\cdot 2}$$
$$Q_2(\theta, \psi, \theta^\cdot, \psi^\cdot)\psi^\cdot \leqslant -\gamma_1\psi^{\cdot 2} \quad (0 < \gamma_1, \ \gamma_2 = \text{const})$$

then under initial conditions for which $r_0=0$ we have $\theta(t) \to 0$ or $\theta(t) \to \pi$, $\psi(t) \to$ const, $\theta^\cdot(t)$, $\psi^\cdot(t)$, $\varphi^\cdot(t) \to 0$ as $t \to \infty$, i.e., each motion asymptotically approaches one of the two equilibrium positions $\theta = 0$ $(\theta = \pi)$, $\theta^\cdot = \psi^\cdot = \varphi^\cdot = 0$.

The authors thank V.V. Rumiantsev for attention to the work and for discussing the results.

REFERENCES

1. LIAPUNOV A.M., General Problem of Stability of Motion. Moscow—Leningrad, GOSTEKHIZDAT, 1950.
2. RUMIANTSEV V.V., On stability of motion relative to a part of the variables. Vestn. Moskovsk. Gos. Univ., Ser. Mat., Mekh., Astron., Fiz., Khim., No.4, 1957. (See also, in English, PMM, Vol.35, No.1, 1971).
3. MATROSOV V.M., On the stability of gyroscopic systems with dissipation. Tr. Kazansk.Aviats. Inst., Vol.45, 1959.
4. MATROSOV V.M., On the stability of gyroscopic systems. Tr. Kazansk. Aviats. Inst., Vol.49, 1959.
5. MERKIN D.P., Gyroscopic Systems. Moscow, NAUKA, 1974.
6. TEREKI I. and KHATVANI L., On partial stability and convergence of motions. PMM Vol.45, No. 3, 1981.
7. BELLMAN R., Stability theory of differential equations. New York—Toronto—London: McGraw—Hill, 1953.
8. ROUCHE N., HABETS P. and LALOY M., Stability theory by Liapunov's direct method. New York—Heidelberg—Berlin: Springer-Verlag, 1977.
9. RUMIANTSEV V.V., On the stability of motion of a gyroscope on gimbals. PMM Vol.22, No.3, 1958.
10. KREMENTULO V.V., Stability of a gyroscope having a vertical axis of the outer ring, with dry friction in the gimbal axes taken into account, PMM Vol.24, No.3, 1960.

Translated by N.H.C.